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LETTER TO THE EDITOR

A new class of exact solutions for coupled scalar field equations

N N Rao† and D J Kaup

Department of Mathematics, Clarkson University, Potsdam, NY 13699, USA

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Abstract. We report a class of exact solutions valid in new regimes of parameter values for a generic system of nonlinear coupled scalar field equations which is the stationary reduction of the Schrödinger–Boussinesq (or Korteweg–de Vries) system, and which extends the Henon–Heiles system to Hamiltonians with indefinite kinetic energy. The results are applied to co-propagating coupled nonlinear waves in magnetized plasmas.

Many physical problems lead to nonlinear differential equations whose exact analytical solutions are of great importance and interest for a variety of theoretical as well as practical reasons [1]. Recently, one of us (NNR) introduced [2] a generic system of coupled scalar field equations dealing with the nonlinear coupling of a high-frequency wave to a suitable low-frequency wave in a dispersive medium. The equations are equivalent to the stationary equations obtainable from the Schrödinger–Boussinesq (or Korteweg–de Vries) system which is known to describe the nonlinear evolution of the modulational instability of high-frequency wave packets in dispersive media leading to solitary wave structures [3, 4]. Furthermore, as discussed below, they extend the parameter regimes for the Henon–Heiles equations [1] to include the case of Hamiltonians with indefinite kinetic energy. Equations similar to the generic system also occur in relativistic quantum field theories in $1+1$ dimensions for localized fields with finite energy [5]. It has been recently shown that the nonlinear Schrödinger as well as the Korteweg–de Vries equations can be obtained as the symmetry reductions of the self-dual Yang–Mills field equations [6, 7]. In earlier work [2], it was shown that the generic equations admit different classes of solutions valid on different hypersurfaces in the allowed parameter space. We report in this letter a class of solutions valid in new regimes of the parameter values.

Consider the generic system of equations [2]

$$\beta \frac{d^2 E}{d\xi^2} = b_1 E + b_2 \Phi E \tag{1}$$

$$\lambda \frac{d^2 \Phi}{d\xi^2} = d_1 \Phi + d_2 \Phi^2 + d_3 E^2 \tag{2}$$

where E and Φ are any two scalar fields, and ξ is the independent variable; λ, β, b_1, b_2, d_1, d_2 and d_3 are parameters; all the quantities occurring in (1) and (2) are real.

† Permanent address: Theoretical Physics Division, Physical Research Laboratory, Navrangpura, Ahmedabad 380009, India.

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Equations (1) and (2) can be obtained from the coupled Schrödinger–Boussinesq (or Korteweg–de Vries) system by going into a stationary frame of reference [2]. They can be derived from the Lagrangian

$$L = \beta d_3 \left(\frac{\mathrm{d}E}{\mathrm{d}\xi}\right)^2 + \tfrac{1}{2}\lambda b_2 \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right)^2 - V(E, \Phi) \tag{3}$$

where the 'potential' $V(E, \Phi)$ is given by

$$V(E, \Phi) = -(d_3 b_1 E^2 + \tfrac{1}{2} b_2 d_1 \Phi^2 + \tfrac{1}{3} b_2 d_2 \Phi^3 + b_2 d_3 \Phi E^2). \tag{4}$$

The corresponding Hamiltonian H is

$$H = \frac{1}{4\beta d_3}(\Pi_E)^2 + \frac{1}{2\lambda b_2}(\Pi_\Phi)^2 + V(E, \Phi) \tag{5}$$

where the generalized momenta Π_E and Π_Φ are defined by

$$\Pi_E = 2\beta d_3 \left(\frac{\mathrm{d}E}{\mathrm{d}\xi}\right) \qquad \Pi_\Phi = \lambda b_2 \left(\frac{\mathrm{d}\Phi}{\mathrm{d}\xi}\right).$$

Using (4) and (5), we eliminate ξ in (1) and (2) to derive the following equation for Φ with respect to E:

$$2\lambda\beta(H - V)\frac{\mathrm{d}^2\Phi}{\mathrm{d}E^2} + \lambda^2 b_2(b_1 E + b_2 \Phi E)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}E}\right)^3 - \lambda b_2 \beta(d_1\Phi + d_2\Phi^2 + d_3 E^2)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}E}\right)^2$$
$$+ 2\lambda\beta d_3(b_1 E + b_2 \Phi E)\left(\frac{\mathrm{d}\Phi}{\mathrm{d}E}\right) - 2\beta^2 d_3(d_1\Phi + d_2\Phi^2 + d_3 E^2) = 0. \tag{6}$$

Equation (6) is complementary to equation (7) of [2].

Equation (6) has a cubic term in the first-order derivative which makes it different from the well known equations of Painlevé type belonging to the polynomial class [8]. While attempts to obtain exact solutions of (6) valid in the entire parameter space have not been successful so far, it has been possible to find a class of solutions valid in new regimes of the parameter values by using

$$\Phi = C_0 + C_1 E \tag{7}$$

where the coefficients C_0 and C_1 are to be determined uniquely and self-consistently. Note that (7) is different from equation (16) of [2] which, as discussed below, led to solutions having different symmetry properties as well as functional dependences on ξ. Using (7), equation (6) can be factorized to yield relations between E and Φ which determine the following sets of parameter values.

Case (A): $C_0 = 0$. For this choice of C_0, the parameters satisfy

$$\lambda b_1 - \beta d_1 = 0 \tag{8}$$

whereas C_1 is given by

$$C_1^2 = \frac{\beta d_3}{\lambda b_2 - \beta d_2} \tag{9}$$

with $\lambda b_2 - \beta d_2 \neq 0$ and $\beta d_3(\lambda b_2 - \beta d_2) > 0$. Both signs of C_1 from (9) are allowed; this is due to the fact that (1) and (2) are invariant under $E \to -E$. Furthermore, any combination of the values of d_1 and d_2 consistent with (8) and (9) is allowed.

Case (B): $d_1+d_2C_0=0$. When d_1, d_2 and C_0 satisfy this relation, the condition between the parameters is

$$\lambda b_1+C_0(\lambda b_2-\beta d_2)=0 \tag{10}$$

As in the previous case, both signs for C_1 from (9) are allowed. There are two cases for C_0:

(i) $C_0=0$ for which the allowed values are either $d_1=0$ and $d_2=0$ or $d_1=0$ and $d_2\neq 0$;

(ii) $C_0\neq 0$ for which case one requires either $d_1=0$, $d_2=0$ and C_0 is determined by (10) or $d_1\neq 0$, $d_2\neq 0$ and $C_0=-d_1/d_2$ in which case (10) becomes

$$\lambda b_1 d_2-d_1(\lambda b_2-\beta d_2)=0 \tag{11}$$

where $\lambda b_2-\beta d_2\neq 0$ is required.

To obtain the explicit solutions, we derive from (7) and (1)

$$\beta\frac{\mathrm{d}^2E}{\mathrm{d}\xi^2}=(b_1+b_2C_0)E+b_2C_1E^2 \tag{12}$$

which can be integrated to yield

$$\left(\frac{\mathrm{d}E}{\mathrm{d}\xi}\right)^2=\frac{2b_2C_1}{3\beta}(E-\theta_1)(E-\theta_2)(E-\theta_3) \tag{13}$$

where θ_1, θ_2 and θ_3 are the roots of the cubic equation

$$\theta^3+\left[\frac{3(b_1+b_2C_0)}{2b_2C_1}\right]\theta^2+\left[\frac{3C}{2b_2C_1}\right]=0 \tag{14}$$

and C is a constant of integration. Equation (13) can be integrated using the transformation

$$Y^2=\frac{E-\theta_2}{\theta_1-\theta_2} \tag{15}$$

to obtain the solution

$$E(\xi)=\theta_1\,\mathrm{sn}^2[\eta(\xi-\xi_0),k]+\theta_2\,\mathrm{cn}^2[\eta(\xi-\xi_0),k] \tag{16}$$

where sn and cn are the Jacobian elliptic functions [8], ξ_0 is a constant of integration, and

$$\eta^2=\frac{b_2C_1}{6\beta}(\theta_3-\theta_2) \tag{17}$$

$$k^2=\frac{\theta_1-\theta_2}{\theta_3-\theta_2}. \tag{18}$$

The solution for $\Phi(\xi)$ is obtained by using (16) in (7), that is

$$\Phi(\xi)=C_0+C_1E(\xi) \tag{19}$$

where C_0 and C_1 are determined as discussed earlier. Note that the solution (16) for $E(\xi)$ is qualitatively different from equation (18) of [9] which deals with a specific example of the generic system (1) and (2).

The constant of integration C determines the wavelength of the periodic solutions (16) and (19). For the special case when $C=0$, the roots of the cubic equation (14) are

$$\theta_1=0 \qquad \theta_2=-\frac{3(b_1+b_2C_0)}{2b_2C_1} \qquad \theta_3=0. \tag{20}$$

Solution (16) together with (18) becomes

$$E(\xi)=-\frac{3(b_1+b_2C_0)}{2b_2C_1}\operatorname{sech}^2[\eta(\xi-\xi_0)] \tag{21}$$

where η is defined by

$$\eta^2=\frac{b_1+b_2C_0}{4\beta}. \tag{22}$$

The corresponding solution for $\Phi(\xi)$ is obtained by using (21) in (19). Both the solutions are symmetric with respect to $\xi=\xi_0$ and, furthermore, have sech^2 dependence on ξ. This is to be contrasted with the solitons reported earlier [2, 4] where $E(\xi)$ was either anti-symmetric having sech tanh dependence or symmetric having sech dependence. In all cases, $\Phi(\xi)$ is always symmetric with sech^2 dependence.

As an application of the above results, we discuss below the coupling between the high-frequency upper-hybrid waves and the low-frequency magnetosonic waves, both of which propagate perpendicular to the external magnetic field in a plasma. The equations governing the stationary wave propagation are [2, 4]

$$D_0\frac{\mathrm{d}^2E}{\mathrm{d}\xi^2}=\lambda E+2\mu\omega_{H0}NE \tag{23}$$

$$\beta^2\frac{\mathrm{d}^2N}{\mathrm{d}\xi^2}=\alpha N-a^2N^2-\eta^2E^2 \tag{24}$$

where E and N are respectively the normalized upper-hybrid electric field amplitude and the plasma number density perturbation; all the symbols in (23) and (24) are defined as in [2, 4]. While (23) is obtained from a Schrödinger-like equation, (24) is the stationary reduction of either the Boussinesq equation or the Korteweg-de Vries equation depending on α. Comparing (23) and (24) with (1) and (2), we write down the explicit solutions for the two cases mentioned earlier.

First, we consider localized solutions for which there are two cases possible:

Case (A). When the parameters satisfy the condition, $\lambda\beta^2-\alpha D_0=0$, that is

$$\frac{\lambda\beta^2}{D_0}=\begin{cases}(M^2-V_{\mathrm{M}}^2) & \text{for Boussinesq equation}\\ 2V_{\mathrm{M}}(M-V_{\mathrm{M}}) & \text{for Korteweg-de Vries equation}\end{cases} \tag{25}$$

the explicit solutions are

$$E(\xi)=\pm\frac{3\lambda}{4\mu\omega_{H0}}\left[-\frac{D_0a^2+2\beta^2\mu\omega_{H0}}{\eta^2D_0}\right]^{1/2}\operatorname{sech}^2[\kappa(\xi-\xi_0)] \tag{26}$$

$$N(\xi)=-\frac{3\lambda}{4\mu\omega_{H0}}\operatorname{sech}^2[\kappa(\xi-\xi_0] \tag{27}$$

where ξ_0 is a constant of integration, and

$$\kappa=\left(\frac{\lambda}{4D_0}\right)^{1/2}. \tag{28}$$

In order that $E(\xi)$ be real, (26) requires $D_0<0$ such that

$$2\beta^2\mu\omega_{H0}-a^2|D_0|>0. \tag{29}$$

From (27) and (28), one requires that $\lambda<0$ and $N>0$. From (25), it is easy to conclude that the coupled stationary waves propagate with super-magnetosonic speed ($M>V_{\mathrm{M}}$) accompanied by compressional ($N>0$) density perturbations. Unlike the earlier solutions [2, 4], both the upper-hybrid wave envelope as well as the magnetosonic density perturbations are symmetric with respect to ξ_0 thereby recovering the solutions found earlier [10].

Case (B). For parameters satisfying

$$\lambda\beta^2a^2+\alpha(D_0a^2+2\beta^2\mu\omega_{H0})=0 \tag{30}$$

the solutions are

$$E(\xi)=\pm E_0\,\mathrm{sech}^2[\kappa(\xi-\xi_0)] \tag{31}$$

$$N(\xi)=-N_0\,\mathrm{sech}^2[\kappa(\xi-\xi_0)] \tag{32}$$

where

$$E_0=\frac{3(\lambda a^2+2\mu\alpha\omega_{H0})}{4\mu\omega_{H0}a^2}\left[-\frac{D_0a^2+2\beta^2\mu\omega_{H0}}{\eta^2D_0}\right]^{1/2} \tag{33}$$

$$N_0=\frac{3\lambda a^2+2\mu\omega_{H0}\alpha}{4\mu\omega_{H0}a^2} \tag{34}$$

and κ is now given by

$$\kappa=\left[\frac{\lambda a^2+2\mu\omega_{H0}\alpha}{4D_0a^2}\right]^{1/2} \tag{35}$$

where $D_0<0$ satisfies (29) so that E_0 is real. Equation (20) then becomes

$$\frac{\lambda\beta^2a^2}{|D_0|a^2-2\beta^2\mu\omega_{H0}}=\begin{cases}(M^2-V_{\mathrm{M}}^2) & \text{for Boussinesq equation}\\ 2V_{\mathrm{M}}(M-V_{\mathrm{M}}) & \text{for Korteweg–de Vries equation.}\end{cases} \tag{36}$$

In order that κ is real, (35) requires $\lambda a^2+2\mu\omega_{H0}\alpha<0$. Substituting for λa^2 we find that the inequality can be satisfied only for $\alpha<0$, that is, $M<V_{\mathrm{M}}$. This shows that in this parameter range only sub-magnetosonic solutions are possible in contrast to the previous case where the solitary wave speed is super-magnetosonic. Using (36) in (34), we find $N_0>0$ which together with (32) shows that the upper-hybrid waves are accompanied by rarefaction density perturbations.

Next, we consider the periodic solutions. In either case, they are obtained by solving the cubic equation (14) which, in the present example, becomes

$$\theta^3+\left[\frac{3(\lambda+2\mu\omega_{H0}C_0)}{4\mu\omega_{H0}C_1}\right]\theta^2+\left[\frac{3C}{4\mu\omega_{H0}C_1}\right]=0. \tag{37}$$

If θ_1, θ_2 and θ_3 are the three real roots of (37), then the periodic solutions are given by (16) and (19) where k is given by (18) and η by

$$\eta^2=\frac{2\mu\omega_{H0}C_1}{6D_0}(\theta_3-\theta_2). \tag{38}$$

The roots are chosen in such a way that (18) and (38) yield real values for k and η and the coefficients C_0 and C_1 are determined as earlier.

It is of interest to point out the relationship between the generic equations, namely, (1) and (2), and the well known Henon–Heiles equations [1, 11]. With some algebra, the number of independent parameters in (1) and (2) can be reduced by a suitable rescaling of the variables leading to

$$\frac{d^2E}{d\xi^2}=p_1E+2\Phi E \tag{39}$$

$$\frac{d^2\Phi}{d\xi^2}=A_1\Phi+A_2\Phi^2+p_2E^2 \tag{40}$$

where $p_1=\pm1$, $p_2=\pm1$ and A_1, A_2 are parameters. The Hamiltonian for (39) and (40) is

$$H=\frac{1}{2p_2}(\Pi_E)^2+\frac{1}{2}(\Pi_\Phi)^2-\left(\frac{1}{2}p_1p_2E^2+\frac{1}{2}A_1\Phi^2+\frac{1}{3}A_2\Phi^3+p_2\Phi E^2\right) \tag{41}$$

where $\Pi_E=p_2(dE/d\xi)$ and $\Pi_\Phi=d\Phi/d\xi$. For $p_2=+1$, the 'kinetic energy' in (41) is positive definite, and equations (39) and (40) with $p_1<0$ and $A_1<0$ correspond to the Henon–Heiles system which is known to be integrable for certain discrete sets of values of A_1 and A_2 [11]. On the other hand, for $p_2=-1$, the Hamiltonian has indefinite kinetic energy, and the system (39), (40) is fundamentally different from the usual equations in classical dynamics where the kinetic energy is positive definite. In fact, as discussed elsewhere [12], the governing equations for the envelope solitons of coupled waves yield precisely such Hamiltonians having indefinite kinetic energy; the application considered above is just one example illustrating this feature. The solutions obtained in the present work as well as those found earlier [2, 4] indicate the integrability of (39) and (40) for some specific values of the free parameters and/or initial (boundary) conditions. It is possible that the generic system can be investigated using the methods applied to the Henon–Heiles and other related systems [1, 11]. However, the question of complete integrability in the entire allowed parameter space as well as the possible existence of chaotic behaviour of Hamiltonian systems with indefinite kinetic energy is a subject not well understood and therefore needs further investigations.

To summarize, we have found a class of exact analytical solutions for a generic system of coupled scalar field equations which are the stationary reduction of the Schrödinger–Boussinesq (Korteweg–de Vries) equations and which extend the parameter regimes of the Henon–Heiles system. The solutions are in general periodic, existing in two different regions of the allowed parameter space, and yield, as a limiting case, localized symmetric solutions for both the fields having sech^2 dependence. We have applied the results to a specific example of coupled wave propagation in magnetized plasmas and have shown the existence of new parameter regimes for localized upper-hybrid and magnetosonic waves.

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